Constructing and Consolidating Mathematical Entities in the Context of Whole-Class Discussion

Thérèse Dooley
St.Patrick's College, Dublin
<therese.dooley@spd.dcu.ie>

In this paper, the RBC+C framework (Hershkowitz, Schwartz, & Dreyfus, 2001) is used to analyse and describe construction and consolidation of mathematical knowledge by primary pupils in a whole-class setting. I describe a lesson that concerned what is commonly termed the Handshakes problem. One pupil spontaneously established a connection with a related problem in which the class had engaged a month previously. There followed a conversation in which an older construct was consolidated while a new construct emerged - the nature of this intertwined construction and consolidation is discussed.

In research related to mathematical abstraction, a theoretical framework that is receiving considerable attention is 'Abstraction in Context' (AiC) (Schwartz, Dreyfus, & Hershkowitz, 2009). The three epistemic or observable actions identified by Hershkowitz and her colleagues (Hershkowitz, et al., 2001; Schwartz, et al., 2009) as giving a strong indication that mathematical abstraction is taking place are 'recognizing' (R), 'building-with' (B) and 'constructing' (C). R and B-actions are nested within C-actions and indeed C-actions might be nested within more global C-actions. Furthermore RBC actions have been found to be distributed when groups of students collaborate. The phase following construction is consolidation during which there is conscious reuse of the new construct for recognising or building-with purposes. For this reason the model is often termed the RBC+C model where the second C refers to consolidation. In this paper I use the RBC+C model to analyse the construction by primary pupils of mathematical knowledge in the context of whole class discussion. The lesson I describe ('Chess') is commonly known as the 'Handshakes' problem. In one phase of the plenary discussion a pupil spontaneously made a connection with a related problem ('Friendship Notes') with which the class had engaged one month previously. In the ensuing interaction a group of pupils built with understandings developed in the earlier lesson to construct new understandings of Chess. I argue that in this instance consolidation was also distributed. Furthermore the teacher has a particular role to play in drawing attention to earlier constructs while at the same time taking care not to disrupt the flow of conversation.

Abstraction in Context

The AiC framework was first proposed by Hershkowitz, Schwartz, and Dreyfus (2001) and has been verified since in a variety of contexts (Schwartz, Dreyfus, & Hershkowitz, 2009). Schwartz et al. define abstraction as "an activity of vertically reorganizing previous mathematical constructs within mathematics and by mathematical needs so as to lead to a construct that is new to the learner" (2009, p.24). They suggest that the genesis of abstraction passes through a three stage process: (i) the need for a new construct, (ii) construction and (iii) consolidation of the construct. The need arises from the design of the problem or the student's motivation to solve a particular problem. The RBC actions are used to model construction, the second phase of the abstraction process. Recognition occurs when a student realizes that specific prior knowledge is appropriate for the mathematical problem with which he/she is dealing. When 'building-with', the student is not enriched with new, more complex structural knowledge but is using available structural knowledge to

In J. Dindyal, L. P. Cheng & S. F. Ng (Eds.), Mathematics education: Expanding horizons (Proceedings of the 35th annual conference of the Mathematics Education Research Group of Australasia). Singapore: MERGA.
© Mathematics Education Research Group of Australasia Inc. 2012

deal with the problem at hand. This stage is evident when he or she is involved in an application task or making a hypothesis or justifying a statement. Constructing, the most significant of the epistemic actions that are constituent of abstraction, is a process of building more complex structures from simpler structures. It involves the reorganization of mathematical elements so that a more refined structure emerges. In order to distinguish between 'building-with' and 'constructing', it helps if the goals of the particular activities are considered. In constructing, students use a new mathematical structure to attain their goal. In 'building-with', a goal is attained by combining existing structures. These three epistemic actions are not linear but nested. In other words, 'recognizing' (R) and 'building with' (B) do not precede the process of 'constructing' (C) but are rather nested within it. Furthermore a construction (or C-action) might have nested within it not only a large number of R- and B- actions but also other C-actions. While the RBC model of abstraction was derived from the study of an individual student (Hershkowitz, et al., 2001) and from one of a pair of students working collaboratively (Dreyfus, Hershkowitz, & Schwartz, 2001), it has since been validated in a variety of contexts. Of particular relevance to this paper is that it has been used to describe a 'collective abstraction process' (Hershkowitz, 2009) where different students contribute different parts to the constructing activity. This has been found in the context of a small group of students (Hershkowitz, Hadas, & Dreyfus, 2006) and also in that of whole class discussion (Dooley, 2007).

The third phase of the abstraction process – consolidation – occurs when the learner uses a construct in a flexible manner, for example, when recognising or building-with the structure or using it as a resource in constructing a new mathematical entity (Tsamir & Dreyfus, 2005). The most frequently observed mechanism of consolidation is that of building-with the construct (Dreyfus, Hadas, Hershkowitz, & Schwartz, 2006). As with constructing actions, consolidation has been observed in students working alone on tasks (Monaghan & Ozmantar, 2006) and in pairs of students collaborating on a problem (Tabach, Hershkowitz, & Schwartz, 2006). It can occur in the immediate aftermath of construction (Monaghan & Ozmantar, 2006) and also after a delay of a few months (Tabach, et al., 2006). Furthermore, bearing the characteristics of immediacy, self-evidence, confidence, flexibility and awareness, it is often associated with the language of certitude, e.g., 'clearly', 'obviously' etc. (Dreyfus and Tsamir, 2004).

Background

I conducted a 'classroom design experiment' (Cobb, Gresalfi, & Hodge, 2009) in three different primary schools in Ireland in order to investigate the construction of new mathematical ideas by pupils (Dooley, 2010). Data collected included field notes, audiotapes of whole-class and group interactions, pupils' written artefacts, digital photographs of blackboard recordings, interviews with teachers and, in two of the schools, pupil diaries and post-lesson interviews with small groups of pupils. Retrospective analysis of data was conducted on micro- (between lessons) and macro-levels (between and after cycles of research in the three classrooms). I developed an analytic framework that comprised four dimensions:

1. Mathematical Principles (the constructs that pupils could be expected to develop by engagement with the task);
2. RBC Epistemic Actions;
3. Vague Language (Hedges and Pronouns); and
4. Teacher Follow-up Moves.

For the purpose of this paper, I will focus mainly to the RBC dimension of the framework. However, some reference will also be made to pupils' use of vague language as such language has been found to be integral to RBC+C actions (Dooley, 2011). The lessons I describe took place in the third cycle of research with a class of 31 pupils aged 9 – 10 years.

The Lessons

'Friendship Notes' reads as follows:

> As part of Friendship Week in Greenville School, each pupil writes a short note to each other pupil in his/her class. Each pupil is given one sheet of paper for each note. How many sheets of paper are needed if there are 5 pupils in a class? 10 pupils? What would be the number for any number of pupils?

The Chess lesson took place four weeks after the Friendships Notes lesson and reads as follows:

> In a chess league each participant plays a game of chess with all other participants. How many games will there be if there are 3 participants? 10 participants? 20? Is there a way to find the number of games for any number of participants?

Both Friendship Notes and Chess are characterised by non-reflexivity (that is, no element of a set relates to itself). The main difference between the activities lies in the property of symmetry. 'Chess' is symmetrical because if A relates to ('competes with') B, then it follows that B relates to A. However, in 'Friendship Notes', if A relates to ('writes to') B, the reciprocal relationship is not implied. For this reason the function mapping n (the number of people) to y (the number of notes) in the Friendships Notes is y = $n(n\text{-}1)$ while in Chess, $y = n(n-1)/2$ where n represents the number of people and y the number of games. In Friendship Notes a number of pupils constructed the formula with reference to its asymmetric, non-reflexive properties. For example, David described the asymmetric aspect:

David: Because you need em like … one person would need to give one to the other and the other person would give one back.

Catherine rationalised a solution for $n = 20$ (i.e., 20 x 19) as follows:

Catherine: Em well you don't give one to yourself so you take away … one.
TD: Yeah
Catherine: And then em you multiply them together.

The Chess lesson took place over two days. At the end of the first session (Chess 1), David, conjectured that the number of games for 20 people might be found by multiplying 20 by 19 and halving the product. He built with this strategy in Chess 2 to state a generalised formula. However, he was unable to verify his proposed solution method with reference to the underlying structure of the problem, that is, to give it 'structural verification' (Rowland, 1999). Another student, Enda, made a connection with Friendship Notes and this led to a new construction, i.e., structural verification of the formula. The transcript[1] below concerns this C-action.

[1] Transcript conventions are: TD: the researcher/teacher (myself); Ch: a child whose name I was unable to identify in recordings; … : a short pause; […]: a pause longer than three seconds; [ ]: lines omitted from transcript because they are extraneous to the substantive content of the lesson.

Turn	Transcription	Pupil Action	Epistemic Action (RBC)	
636	David: Multiply it by the number less …			Constructing
637	TD: Huh, huh.			
638	David: […] and then half it.	David proposed a generalised formula for Chess.	Building-with	
639	Enda: It looks like … it's pretty much the very same as the friendship cards, it seems kind of like that.	Enda made a connection with 'Friendship Notes'.	Building-with	
640	TD: Right, Enda, do you remember the friendship notes, that's a good thing. Do you remember the friendship notes? Do you remember what you did for the friendship notes? What did you do for the friendship notes? Do you remember the rule? Barry?			
641	Barry: It's kind of the same thing as, eh, you wouldn't have to do themselves so there's going to be one less.	Barry referred to the non-reflexive nature of both activities.	Building-with	
642	TD: Ok, so but the rule … according to David, when we were doing the friendship notes, [ ] For example in friendship notes if there were three children than how many notes would there be for three children? … Do you remember? … Right, Barry? Does anyone remember how many notes there were for three children in the friendship notes? … Yeah?			
643	Colin: Em, six.	Colin recalled number of friendship notes for six children.	Recognising	

644	TD: Six but see in the chess game it's only three. So why is it a bit different? Does anyone know why it's a bit different? […] Myles?			
645	Myles: Em because in chess you will just have to play them, if they played you one time then you have kind of played them once.	Myles referred to symmetric nature of 'Chess'.	Building-with	
646	Ch: Ah!			
647	Myles: In friendship notes you have to play them kind of again so like you give them your note and they will have to … they will still give you back a note.	Myles referred to the asymmetric nature of 'Friendship Notes'.		
648	TD: Ok, so once you play the game you don't play it back, isn't that what you are saying?			
649	Myles: Yeah.			
650	TD: That's what you are saying. Yes?			
651	Colin: Em, well cos in the friendship notes you have to give two because if there were three you would have to give one to each person …	Colin referred to the asymmetric nature of 'Friendship Notes'.	Building-with	
652	TD: Hm, hm.			
653	Colin: … and everyone has to give one to each person, so it's the same as three by two.			
654	TD: Hm, hm.			
655	Colin: Eh, and in chess you only have to play them once even if they challenge you.	Colin referred to symmetric nature of 'Chess'.		
656	TD: Hm, hm.			
657	Colin: So eh …			
658	TD: And what does that mean for the chess game then? What does it mean for the chess … rule?			
659	Colin: Eh, you don't … you don't play them twice.			
660	Ch: Ah!			

661	TD: Ok, so what happens then, what's the rule for the chess? Enda?			
662	Enda: Eh well, I actually definitely agree with David's way by doing the friendship notes, the same way as the friendship notes and halving it …	Enda made a connection between David's formula and 'Friendship Notes'.	Building-with	
663	TD: Hm, hm.			
664	Enda: … because all of the things we get in that are half what we get in the chess thing.			
665	TD: Hm, hm.		Building-with	
666	Enda: So I definitely agree with David's way by multiplying by one number less and halving it. I definitely agree with that now.			

Epistemic Actions

Enda's input in turn 639 above is characterised by vagueness (e.g., " … it seems kind of like that."). His use of the adaptor, 'kind of', for example, allows him to make an assertion without fully committing to it and thus shield himself from accusation of error (Rowland, 2000), an important consideration given his large audience. I, for obvious reasons, greet his suggestion with enthusiasm and, following my probing questions, Barry endeavours to explicate the connection. Again he is tentative about the connection between the two problems ("It's kind of the same thing …) but his description of the non-reflexive nature of both problems is characterised by more certainty (i.e., "you wouldn't have to do themselves so there's going to be one less"). Myles then describes quite confidently the structure of Chess (turn 645). It would seem that his grasp of the asymmetric structure of Friendship Notes (expressed in turn 647) facilitates his understanding of the symmetric structure of Chess. Colin reasoning in turns 651, 653 and 655 is similar. Moreover, in turn 659, he builds-with further by intimating the multiplicative relationship between Chess and Friendship (" [In Chess] you don't play them twice"). According to Monaghan and Ozmantar (2006), consolidation of a previously constructed entity is evidenced by the establishment of interconnections between it and a new construction and by reasoning with these constructions. At the beginning of the constructing action above, Enda was uncertain about the nature of the relationship between Chess and Friendship Notes. However, the reasoning in which Barry, Myles and Colin engaged and their consolidation of the previous construct establishes the connection for him and indeed it is he, Enda, who expresses this relationship in turns 662, 664 and 666. Here his input is marked by certainty and confidence. In turn 664, he finds support for David's suggested method in the list of solutions for Friendship Notes which was available in his diary ("because all of the things we get in that are half what we get in the chess thing"). The error he makes (since the Friendship Notes solutions are double those for Chess!) is obviously a slip. Underpinning the strategy that he states in turn 666 ("multiplying by one number less and halving it") are the non-reflexive and symmetrical aspects of the problem explicated by some of his peers.

Concluding Remarks

According to Tabach, et al. (2006) knowledge constructing and consolidating are dialectical processes in that "new *Constructs* stem from old ones already *Consolidated*, which gain *Consolidating* through the new *Construction*, creating a new abstract entity" (p.255, italics in original). This dialectical process is evident in the transcript above as students move back and forth between consolidation of the Friendship Notes construction and development of a new entity (structural verification of the Chess problem). As they build-with the Friendship Notes construction, they consolidate it further. This is evidenced, in particular, by the generalised manner in which they speak of the construct – in particular the pronoun 'you' functions as a 'vague generalizer' (Rowland, 2000) suggesting that the pupils are detaching from specific examples to a more generalised description of the structure of the (Friendship Notes) problem. This consolidation, which is used for building-with, is nested within the C-action. Furthermore, it allows the pupils concerned to refer also to Chess in generalities (e.g., "you wouldn't have to do themselves so there's going to be one less" and "you don't play them twice"). There is also evidence that just as constructing can be distributed, so too is consolidation. This is possibly the case because the students 'reconstruct' the Friendship Notes construct in order to verify structurally the formula for Chess. The reconstruction is not, as maintained by Monaghan and Ozmantar (2006), the same as consolidation but rather an important part of it. Nonetheless, consolidation of the Friendship Notes structure (as expressed, for example, in turns 641, 647 and 653) is distributed among a few of the class members.

In previous papers I have described how whole-class conversation can be a vehicle for the construction of new mathematical entities. What is critical is the creation of a 'conjecturing atmosphere' in which pupils try out, test and modify ideas (Mason, 2008). Over direction by the teacher can stifle such an atmosphere particularly if pupils feel that their ideas are being evaluated. In the interaction above I as teacher-researcher might be accused of 'over-directing' children's contributions. However, the first mention of Friendship Notes was made spontaneously by Enda. Ozmantar and Monaghan (2007) have suggested that "if students are not aware of the importance and necessity of the knowledge artefacts available to them for a new construction, they are unlikely to draw upon them as they (or their attention) are blocked" (p.106). It is for this reason that I broadcast Enda's contribution to the class and then set sub goals (e.g., in turns 640, 642 and 658). However, in the context of whole-class discussion, such interventions by the teacher have to be used with care since they might serve to disenfranchise other students and curb contributions that they could make to the construction process. Indeed although there were 31 pupils in the class, the transcript above concerns only five of them. In subsequent written reflections, many pupils aligned themselves with David's solution method. Furthermore, other pupils developed different constructs in that lesson and still others built-with some of the constructs developed in Chess in a related lesson that took place on the following day. This suggests that while some pupils explicitly consolidated and contributed to the construction of new entities there were others in the class who engaged tacitly in the process. Further analysis is required to investigate the extent of this.

References

Cobb, P., Gresalfi, M., & Hodge, L. L. (2009). A design research perspective on the identity that students are developing in mathematics classrooms. In B. Schwartz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 223 - 243). London and New York: Routledge.

Dooley, T. (2007). Construction of knowledge by primary pupils: The role of whole-class interaction. In D. Pitta-Pantazi & G. Philippou (Eds.), *Fifth congress of the European Society for Research in Mathematics Education* (pp. 1658 - 1667). Larnaca, Cyprus: CERME.

Dooley, T. (2010). *The construction of mathematical insight by pupils in whole-class conversation* (Unpublished doctoral dissertation). University of Cambridge, Cambridge.

Dooley, T. (2011). RBC epistemic actions and the role of vague language. In B. Ubuz (Ed.), *Proceedings of the 35th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 281 - 288). Ankara, Turkey: PME.

Dreyfus, T., Hadas, N., Hershkowitz, R., & Schwartz, B. (2006). Mechanisms for consolidating knowledge constructs. In N. J., H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 465 - 472). Prague: PME.

Dreyfus, T., Hershkowitz, R., & Schwartz, B. (2001). Abstraction in context II: The case of peer interaction. *Cognitive Science Quarterly*, *1*(3/4), 307 - 368.

Dreyfus, T., & Tsamir, P. (2004). Ben's consolidation of knowledge structures about infinite sets. *The Journal of Mathematical Behavior*, *23*(3), 271 - 300.

Hershkowitz, R. (2009). Contour lines between a model as a theoretical framework and the same model as methodological tool. In B. Schwartz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 273 - 280). London and New York: Routledge.

Hershkowitz, R., Hadas, N., & Dreyfus, T. (2006). Diversity in the construction of a group's shared knowledge. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds.), *Proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 297 - 304). Prague: PME.

Hershkowitz, R., Schwartz, B., & Dreyfus, T. (2001). Abstraction in context: Epistemic actions. *Journal for Research in Mathematics Education*, *32*(2), 195 - 222.

Mason, J. (2008). Making use of children's powers to produce algebraic thinking. In J. J. Kaput, D. W. Carraher & M. L. Blanton (Eds.), *Algebra in the early years* (pp. 57 - 94). New York and London: Lawrence Erlbaum Associates.

Monaghan, J., & Ozmantar, M. (2006). Abstraction and consolidation. *Educational Studies in Mathemati*cs, *62*, 233 - 258.

Ozmantar, M., & Monaghan, J. (2007). A dialectical approach to the formation of mathematical abstractions. *Mathematics Education Research Journal*, *19*(2), 89 - 112.

Rowland, T. (1999). 'i' is for induction. *Mathematics Teaching*, *167*, 23 - 27.

Rowland, T. (2000). *The pragmatics of mathematics education: Vagueness in mathematical discourse*. London: Falmer Press.

Schwartz, B., Dreyfus, T., & Hershkowitz, R. (2009). The nested epistemic actions model for abstraction in context. In B. Schwartz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 11 - 41). London and New York: Routledge.

Tabach, M., Hershkowitz, R., & Schwartz, B. (2006). Constructing and consolidating of algebraic knowledge within dyadic processes: A case study. *Educational Studies in Mathematics*, *63*, 235 - 258.

Tsamir, P., & Dreyfus, T. (2005). How fragile is consolidated knowledge? Ben's comparisons of infinite sets. *Journal of Mathematical Behaviour*, *24*, 15 - 38.